

A Counter-example to the Cancellation Problem for the Affine Space \mathbb{A}^3 in characteristic p

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Abstract

We show that the Cancellation Problem has a negative solution for the affine space \mathbb{A}_k^3 over any field k of positive characteristic. We prove that an example of Asanuma provides a three-dimensional k -algebra A for which $A[X]$ is isomorphic to $k[X_1, X_2, X_3, X_4]$ but A is not isomorphic to $k[X, Y, Z]$.

1 Introduction

Let k be an algebraically closed field and A be an affine k -algebra. The famous Cancellation Problem for Affine Spaces asks: if $A[X]$ is isomorphic to the polynomial ring $k[X_1, \dots, X_{n+1}]$, does it follow that A is isomorphic to $k[X_1, \dots, X_n]$? For $n = 1$, a positive solution to the problem was given by Abhyankar, Eakin and Heinzer ([1]). For $n = 2$, a positive solution to the problem was given by Fujita, Miyanishi and Sugie ([6], [7]) in characteristic zero and Russell ([8]) in positive characteristic; a simplified algebraic proof was given by Crachiola and Makar-Limanov in [5]. The problem remained open for $n > 2$.

In this paper, we shall show that for $n = 3$, an example of T. Asanuma discussed in [2] and [3] gives a negative solution to the Cancellation Problem in positive characteristic (Corollary 3.5). For convenience, we shall use the notation $R^{[n]}$ for a polynomial ring in n variables over a ring R .

In [2], Asanuma has proved the following theorem (cf. [2, Theorem 5.1, Corollary 5.3], [3, Theorem 1.1]).

Theorem 1.1. *Let k be a field of characteristic $p(> 0)$ and*

$$A = k[X, Y, Z, T]/(X^m Y + Z^{p^e} + T + T^{sp})$$

where m, e, s are positive integers such that $p^e \nmid sp$ and $sp \nmid p^e$. Let x denote the image of X in A . Then A satisfies the following two properties:

1. $A^{[1]} \cong_{k[x]} k[x]^{[3]} \cong_k k^{[4]}$.

2. $A \not\cong_{k[x]} k[x]^{[2]}$.

In [3, Theorem 2.2], Asanuma used the above example to construct non-linearizable algebraic torus actions on \mathbb{A}_k^n over any infinite field k of positive characteristic when $n \geq 4$. The following problem occurs in the same paper (see [3, Remark 2.3]).

Question. Let A be as in Theorem 1.1. Is $A \cong_k k^{[3]}$?

In this paper we shall use techniques developed by L. Makar-Limanov and A. Crachiola to show (Theorem 3.4) that the answer to the above problem is negative when $m > 1$ and $p \nmid m$. Thus, in view of Theorem 1.1, we get counter-examples to the Cancellation Problem in positive characteristic for $n = 3$ (Corollary 3.5).

2 Preliminaries

Definition. Let A be a k -algebra and let $\phi : A \rightarrow A^{[1]}$ be a k -algebra homomorphism. For an indeterminate U over A , let the notation ϕ_U denote the map $\phi : A \rightarrow A[U]$. ϕ is said to be an *exponential map* on A if ϕ satisfies the following two properties:

- (i) $\varepsilon_0 \phi_U$ is identity on A , where $\varepsilon_0 : A[U] \rightarrow A$ is the evaluation at $U = 0$.
- (ii) $\phi_V \phi_U = \phi_{V+U}$, where $\phi_V : A \rightarrow A[V]$ is extended to a homomorphism $\phi_V : A[U] \rightarrow A[V, U]$ by setting $\phi_V(U) = U$.

The ring of ϕ -invariants of an exponential map ϕ on A is a subring of A given by

$$A^\phi = \{a \in A \mid \phi(a) = a\}.$$

An exponential map ϕ is said to be non-trivial if $A^\phi \neq A$. We define the Derksen invariant of A , denoted by $\text{DK}(A)$, to be the subring of A generated by the A^ϕ s, where ϕ varies over the set of non-trivial exponential maps on A .

We summarise below some useful properties of an exponential map ϕ .

Lemma 2.1. *Let A be an affine domain over a field k . Suppose that there exists a non-trivial exponential map ϕ on A . Then the following holds:*

- (i) A^ϕ is factorially closed in A , i.e., if $a, b \in A$ such that $0 \neq ab \in A^\phi$, then $a, b \in A^\phi$.
- (ii) A^ϕ is algebraically closed in A .
- (iii) $\text{tr. deg}_k(A^\phi) = \text{tr. deg}_k(A) - 1$.
- (iv) There exists $c \in A^\phi$ such that $A[c^{-1}] = A^\phi[c^{-1}]^{[1]}$.
- (v) If $\text{tr. deg}_k(A) = 1$ then $A = \bar{k}^{[1]}$, where \bar{k} is the algebraic closure of k in A and $A^\phi = \bar{k}$.
- (vi) Let S be a multiplicative subset of $A^\phi \setminus \{0\}$. Then ϕ extends to a non-trivial exponential map $S^{-1}\phi$ on $S^{-1}A$ by setting $(S^{-1}\phi)(a/s) = \phi(a)/s$ for $a \in A$ and $s \in S$. Moreover, the ring of invariants of $S^{-1}\phi$ is $S^{-1}(A^\phi)$.

Proof. Statements (i) –(iv) occur in [4, p. 1291–1292]; (v) follows from (ii), (iii) and (iv); (vi) follows from the definition. \square

Let A be an affine domain over a field k . A is said to have a proper \mathbb{Z} -filtration if there exists a collections of k -linear subspaces $\{A_n\}_{n \in \mathbb{Z}}$ of A satisfying

- (i) $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{Z}$,
- (ii) $A = \bigcup_{n \in \mathbb{Z}} A_n$,
- (iii) $\bigcap_{n \in \mathbb{Z}} A_n = (0)$ and
- (iv) $(A_n \setminus A_{n-1}) \cdot (A_m \setminus A_{m-1}) \subseteq A_{n+m} \setminus A_{n+m-1}$ for all $n, m \in \mathbb{Z}$.

Given a proper \mathbb{Z} -filtration on A , $\text{gr}(A)$ is a \mathbb{Z} -graded domain defined by $\text{gr}(A) := \bigoplus_i A_i/A_{i-1}$ and there exists a canonical map $\rho : A \rightarrow \text{gr}(A)$ defined by $\rho(a) = a + A_{n-1}$, if $a \in A_n \setminus A_{n-1}$.

The following version of a result Derksen, Hadas and Makar-Limanov is presented in [4, Theorem 2.6].

Theorem 2.2. *Let A be an affine domain over a field k with a proper \mathbb{Z} -filtration. Let ϕ be a non-trivial exponential map on A . Then ϕ induces a non-trivial exponential map $\bar{\phi}$ on $\text{gr}(A)$. Moreover, $\rho(A^\phi) \subseteq \text{gr}(A)^{\bar{\phi}}$.*

Lemma 2.3. *Let A be an affine k -algebra. If $\text{DK}(A) \subsetneq A$, then A is not a polynomial ring over k .*

Proof. Suppose that $A = k[X_1, \dots, X_n]$, a polynomial ring in n variables over k . For each i , $1 \leq i \leq n$, consider the exponential map $\phi_i : A \rightarrow A[U]$ defined by $\phi_i(X_j) = X_j + \delta_{ij}U$, where $\delta_{ij} = 1$ if $j = i$ and zero otherwise. Then $A^{\phi_i} = k[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$. It follows that $\text{DK}(A) = A$. Hence the result. \square

3 Main Theorem

In this section we will prove our main result Theorem 3.4. We first prove a few elementary lemmas.

Let F be a field with algebraic closure \bar{F} . An F -algebra B is said to be *geometrically integral* if $B \otimes_F \bar{F}$ is an integral domain. It is easy to see that if an F -algebra B is geometrically integral then F is algebraically closed in B . We therefore have the following lemma.

Lemma 3.1. *Let F be a field and B a geometrically integral F -algebra such that $\text{tr. deg}_F B = 1$. If B admits a non-trivial exponential map then $B = F^{[1]}$.*

Proof. Follows from Lemma 2.1 (v) and the fact that F is algebraically closed in B . \square

Lemma 3.2. (i) *Let F be a field of characteristic p and U, V be two indeterminates over F . Let $g = U^n - \alpha V^m + \beta$, where $0 \neq \alpha, 0 \neq \beta \in F$, m and n are positive integers such that m is coprime to p . Then $B = F[U, V]/(g)$ is a geometrically integral F -algebra.*

(ii) *Let B be as above and Z be an indeterminates over B . Let $C = B[Z]/(Z^{p^e} - \theta U^r + \vartheta)$, where $0 \neq \theta, \vartheta \in F$, and e, r are positive integers such that $p \nmid r$. Then C is a geometrically integral F -algebra.*

(iii) B does not admit any non-trivial exponential map when $m > 1$ and $n > 1$, and C does not admit any non-trivial exponential map when $r > 1$.

Proof. Without loss of generality we may assume that F is an algebraically closed field.

(i) Since m is coprime to p , the polynomial $\alpha V^m - \beta$ is square-free in $F[V]$ and hence, by Eisenstein's criterion, g is an irreducible polynomial in $F[V][U]$. Thus B is an integral domain.

(ii) Set $L := F(U)[V]/(\alpha V^m - U^n - \beta)$, the field of fractions of B . Since $p \nmid m$ ($= [L : F(U)]$) and $p \nmid r$, it is easy to see that $Z^{p^e} - \theta U^r + \vartheta$ is prime in $L[Z]$. Since $Z^{p^e} - \theta U^r + \vartheta$ is a monic polynomial, it follows that $(Z^{p^e} - \theta U^r + \vartheta)B[Z]$ is a prime ideal of $B[Z]$. Thus C is an integral domain.

(iii) If $m > 1$ and $n > 1$ then B is not a polynomial ring over F and if $r > 1$ then C is not a polynomial ring over F . Hence the result follows from Lemma 3.1. \square

We now come to the main technical result of this paper.

Proposition 3.3. *Let k be an algebraically closed field of characteristic p (> 0) and*

$$A = k[X, Y, Z, T]/(X^m Y + Z^{p^e} + T + T^{sp}),$$

where m, e, s are positive integers such that $p^e \nmid sp$, $sp \nmid p^e$, $m > 1$ and $p \nmid m$. Let ϕ be a non-trivial exponential map on A . Then $A^\phi \subseteq k[x, z, t]$, where x, z, t, y denote the images of X, Z, T, Y in A . In particular, $\text{DK}(A) \subsetneq A$.

Proof. We first note that A is a \mathbb{Z} -graded algebra over k with the following weights assigned to the generators of A :

$$\text{wt}(x) = -1, \quad \text{wt}(y) = m, \quad \text{wt}(z) = 0, \quad \text{wt}(t) = 0.$$

This grading defines a proper \mathbb{Z} -filtration on A such that $\text{gr}(A)$ is isomorphic to A . We identify A with $\text{gr}(A)$. For each $g \in A$, let \hat{g} denote the image of g under the composite map $A \rightarrow \text{gr}(A) \cong A$. Thus \hat{g} is the homogeneous component of g in A of maximum degree. By Theorem 2.2, ϕ induces a non-trivial exponential map $\bar{\phi}$ on $A(\cong \text{gr}(A))$ and $\hat{g} \in A^{\bar{\phi}}$ whenever $g \in A^\phi$. Using the relation $x^m y = -(z^{p^e} + t + t^{sp})$ if necessary, we observe that for $g \in A$, either

$$\hat{g} = x^i g_1(z, t) \quad \text{for some } i \geq 0 \quad \text{and} \quad g_1(z, t) \in k[z, t] \quad (1)$$

or

$$\hat{g} = x^i y^j g_1(z, t) \quad \text{for some } 0 \leq i < m, j > 0 \quad \text{and} \quad g_1(z, t) \in k[z, t]. \quad (2)$$

We now prove the result by contradiction. Suppose, if possible, that $A^\phi \not\subseteq k[x, z, t]$. Let $f \in A^\phi \setminus k[x, z, t]$. Then, by (1) and (2), $\hat{f} = x^a y^b f_1(z, t) (\in A^{\bar{\phi}})$ for some $0 \leq a < m$, $b > 0$ and $f_1(z, t) \in k[z, t]$. Since A^ϕ is factorially closed in A (cf. Lemma 2.1 (i)), it follows that $y \in A^{\bar{\phi}}$.

We now see that $h(z, t) \in A^{\bar{\phi}}$ for some polynomial $h(z, t) \in k[z, t] \setminus k$. If $f_1(z, t) \notin k^*$, then, as $f_1(z, t) \mid \hat{f}$, we have $h(z, t) := f_1(z, t) \in A^{\bar{\phi}}$ by Lemma 2.1 (i). Now suppose $f_1(z, t) \in k^*$ and $a > 0$. Then, again by Lemma 2.1 (i), $x \in A^{\bar{\phi}}$. Hence, $h(z, t) := z^{p^e} + t + t^{sp} = -x^m y \in$

$A^{\bar{\phi}}$. Next suppose that $f_1(z, t) \in k^*$ and $a = 0$. Then $\hat{f} \in k[y]$. Since $\text{tr.deg}_k(A^{\phi}) = 2$, one can see that there exists $g \in A^{\phi}$ such that $\hat{g} \notin k[y]$. Now replacing f by fg , we get the required $h(z, t)$ (as above). Thus $k[y, h(z, t)] \subseteq A^{\bar{\phi}}$ for some $h(z, t) \in k[z, t] \setminus k$.

Write the integer s as $s'p^r$, where $p \nmid s'$. Since $p^e \nmid sp$, we have $e - r - 1 > 0$ and since $sp \nmid p^e$, we have $s' > 1$. We now define another proper \mathbb{Z} -filtration on A by assigning the following weights to its generators:

$$\text{wt}(x) = -s'p^e(p-1), \quad \text{wt}(y) = ms'p^{e+1}, \quad \text{wt}(z) = ms', \quad \text{wt}(t) = mp^{e-r-1}.$$

Consider $\text{gr}(A)$ with respect to the above filtration. For $g \in A$, let \bar{g} denote the image of g in $\text{gr}(A)$. In $\text{gr}(A)$, $\bar{x}^m \bar{y}$, \bar{z}^{p^e} and \bar{t}^{sp} are homogeneous elements of degree $ms'p^e$ which is greater than mp^{e-r-1} , the degree of \bar{t} . Therefore, since $x^m y + z^{p^e} + t + t^{sp} = 0$ in A , we have $\bar{x}^m \bar{y} + \bar{z}^{p^e} + \bar{t}^{sp} = 0$ in $\text{gr}(A)$. Moreover, it is easy to see that \bar{x} , \bar{y} , \bar{z} and \bar{t} are algebraically independent over k . As $\text{gr}(A)$ is generated by \bar{x} , \bar{y} , \bar{z} and \bar{t} , it follows that

$$\text{gr}(A) \cong k[X, Y, Z, T]/(X^m Y + Z^{p^e} + T^{sp}).$$

Again by Theorem 2.2, $\bar{\phi}$ induces a non-trivial exponential map $\tilde{\phi}(= \bar{\bar{\phi}})$ on $\text{gr}(A)$ and $k[\bar{y}, \bar{h}(z, t)] \subseteq \text{gr}(A)^{\tilde{\phi}}$. Since k is an algebraically closed field, we have

$$\overline{h(z, t)} = \theta \bar{z}^i \bar{t}^j \prod (\bar{z}^{p^{e-r-1}} + \mu_{\ell} \bar{t}^{s'}),$$

for some $\theta, \mu_{\ell} \in k^*$.

We show that $i = j = 0$. If $i \neq 0$, then $\bar{z} \in \text{gr}(A)^{\tilde{\phi}}$. Now $k[\bar{y}, \bar{z}]$ is a subring of $\text{gr}(A)^{\tilde{\phi}}$ and $\text{tr.deg}_k(k[\bar{y}, \bar{z}]) = 2$. By Lemma 2.1 (vi), $\tilde{\phi}$ induces a non-trivial exponential map on

$$\text{gr}(A) \otimes_{k[\bar{y}, \bar{z}]} k(\bar{y}, \bar{z}) \cong \frac{k(Y, Z)[X, T]}{(YX^m + T^{sp} + Z^{p^e})},$$

which contradicts Lemma 3.2 (iii). Similarly $j = 0$.

Since $h(z, t) \notin k$, it follows that there exists $\mu \in k^*$ such that $\bar{z}^{p^{e-r-1}} + \mu \bar{t}^{s'}$ is a factor of $\overline{h(z, t)}$. Set $w := \bar{z}^{p^{e-r-1}} + \mu \bar{t}^{s'}$. Then, by Lemma 2.1 (i), $w \in \text{gr}(A)^{\tilde{\phi}}$.

Suppose, if possible, that $\mu = 1$. Then $w^{p^{r+1}} = (\bar{z}^{p^{e-r-1}} + \bar{t}^{s'})^{p^{r+1}} = -\bar{x}^m \bar{y} \in \text{gr}(A)^{\tilde{\phi}}$, which implies that $\bar{x} \in \text{gr}(A)^{\tilde{\phi}}$ (cf. Lemma 2.1 (i)). Let L be the field of fractions of $C := k[\bar{x}, \bar{y}, w] (\subseteq \text{gr}(A)^{\tilde{\phi}})$. By Lemma 2.1 (vi), $\tilde{\phi}$ induces a non-trivial exponential map on

$$\text{gr}(A) \otimes_C L = k(\bar{x}, \bar{y}, w)[\bar{z}, \bar{t}] \cong \frac{L[Z, T]}{(Z^{p^{e-1-r}} + T^{s'} - w)},$$

which contradicts Lemma 3.2 (iii). Hence $\mu \neq 1$.

Let F be the field of fractions of $E := k[\bar{y}, w] (\subseteq \text{gr}(A)^{\tilde{\phi}})$. Now

$$0 = \bar{x}^m \bar{y} + \bar{z}^{p^e} + \bar{t}^{sp} = \bar{x}^m \bar{y} + (\bar{z}^{p^{e-1-r}} + \mu \bar{t}^{s'})^{p^{r+1}} + \lambda \bar{t}^{sp} = \bar{x}^m \bar{y} + \lambda \bar{t}^{sp} + w^{p^{r+1}}$$

where $\lambda = 1 - \mu^{p^{r+1}} \neq 0$. By Lemma 2.1 (vi), $\tilde{\phi}$ induces a non-trivial exponential map on

$$\text{gr}(A) \otimes_E F = F[\bar{x}, \bar{t}, \bar{z}] \cong \frac{F[X, T, Z]}{(\bar{y}X^m + \lambda T^{sp} + w^{p^{r+1}}, Z^{p^{e-1-r}} + \mu T^{s'} - w)},$$

which again contradicts Lemma 3.2 (iii).

Hence the result follows. \square

Theorem 3.4. *Let k be any field of characteristic $p(> 0)$ and $A = k[X, Y, Z, T]/(X^m Y + Z^{p^e} + T + T^{sp})$, where m, e, s are positive integers such that $p^e \nmid sp$, $sp \nmid p^e$, $m > 1$ and $p \nmid m$. Then $A \not\cong_k k^{[3]}$.*

Proof. Without loss of generality we may assume that k is algebraically closed. Then the result follows from Lemma 2.3 and Proposition 3.3. \square

Corollary 3.5. *The Cancellation Problem does not hold for the polynomial ring $k[X, Y, Z]$, when k is a field of positive characteristic.*

Proof. Follows from Theorem 1.1 and Theorem 3.4. \square

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